SOME LINKS BETWEEN GAME THEORY AND DECISION THEORY IN ECONOMICS

Dominika Crnjac[1], Goran Martinović[2]

[1]Faculty of Electrical Engineering, J.J. Strossmayer University of Osijek, Croatia,
dominika.crnjac@etfos.hr
[2]Faculty of Electrical Engineering, J. J. Strossmayer University of Osijek, Croatia,
goran.martinovic@etfos.hr

Abstract

Certain optimal strategies based upon game theory are given in this paper. A decision-making function and a risk function are explained. Decision-making criteria are applied for determining best decision-making functions with respect to a specific criterion. Special attention is given to the minimax criterion.

JEL classification: C02, C71, D81,

Key words: saddle-point, minimax criterion, Bayesian criterion, loss function, risk function, decision-making function

Introduction

In making a decision or a choice it seems rational to choose an option which has mathematical expectation "which promises most ", i.e. to choose an option which minimizes the expected loss or maximizes the expected gain. Unfortunately, this simple approach to decision-making does not always function, since it is often very difficult to assign a numerical value either to a result or to probability of the outcome. This paper considers game theory and statistical games, and can be used as an introduction to decision theory.

1. Game theory

1.1 Zero-sum games

The game is said to be a zero-sum game when there is a conflict between players, and whatever one player loses in the game the other player wins. It is possible to consider non-zero-sum games for more than two players, but they go beyond the scope of this paper. In game theory it is assumed that while

selecting his/her strategy none of the players knows what the other player will do, noting that once the strategy is selected, it cannot be changed.
Games are classified according to the number of strategies available to every player. For example, if every player has to choose between two strategies, then it is going to be a 2×2 game. If one player has 4 options, and the other 5, then it will be a 4×5 or a 5×4 game.
In this paper we consider just those games in which every player has only finitely many options.

1.1.1 Notation

Let us denote two players by Player A and Player B.
Player A options will be denoted by I, II, III, etc.
Player B options will be denoted by 1, 2, 3, etc.
The game might be presented by a matrix, e.g. a 2×2 game might be presented as follows:

	Player A: I	II
Player B	*L*(I,1)	*L*(II,1)
	L(I,2)	*L*(II,2)

Matrix elements are called payoffs (i.e. loss for Player A and gain for Player B). Amounts *L*(I,1), *L*(II,1), *L*(I,2) and *L*(II,2) are loss function values of a certain game. *L*(I,1) represents loss of Player A (and gain of Player B) when Player A chooses option I, and Player B chooses option 1, etc.
Game theory tries to find optimal strategies, i.e. the ones most profitable for every player, as well as a corresponding payoff or value of the game.

1.1.2 Saddle-point

A pair of strategies will be in balance if and only if element *L*(a,b) corresponding to balance is at the same time the greatest in the column and the least in the row. Such strategy is called a saddle-point.
For example, a 3×3 matrix

		Player A: I	II	III
Player B	1	6	2	2
	2	4	3	5
	3	-3	1	0

has a saddle-point in the second row and the second column, but a 2×2 matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

does not have saddle-points.

1.1.3 Domination

In a payoff matrix one strategy (represented by a row or a column) dominates[14] over the other strategy if the choice of the first strategy is at least as good as the choice of the second strategy and in some cases even better. Player A will look for a lower loss, whereas Player B will look for a greater gain. The dominated strategy can always be rejected in the game.

For example, in a (4×3) game

Player A

Player B	I	II	III	IV
1	2	0	1	4
2	1	2	1	4
3	4	1	3	2

Since Player A looks for a lower loss, strategy II dominates over strategy IV, since the loss of Player A will always be less if strategy II is selected and not strategy IV. Therefore, Player A will never choose strategy IV; hence, it can be rejected.

The game is reduced to a 3×3 game

Player A

Player B	I	II	III
1	2	0	1
2	1	2	1
3	4	1	3

However, since Player B looks for a greater gain, strategy 3 now dominates over strategy 1, since the gain of Player B is always greater if strategy 3 is selected and not strategy 1. Thus, the game is reduced to a 3×2 game

Player A

Player B	I	II	III
2	1	2	1
3	4	1	3

[14] R.B.Myerson; (1997), p.26, Game Theory Analysis of Conflict, First Harvard University Press (paperback edition)

2. Statistical games

By observing social, physical and other phenomena, regulating laws or relations are called state of nature. The conclusion about the state of nature is reached on the basis of results obtained from a statistical experiment. Drawing a conclusion about the state of nature on the basis of experimental results is called statistical inference. A statistical inference problem can be considered a two-player game.

In statistical inference decisions on the population, such as expectation or variance of some property, are based upon sample data. Statistical inference might therefore be observed as a game between the Nature that controls relevant features of the population and a statistician who tries to make a decision on the population.

One way in which statistical games differ from game theory is that in game theory every player selects a strategy without knowing what the opponent will do. In a statistical game, a statistician has sample data which provide some information about the choice of nature.

The state of nature denoted by θ is unknown to a statistician. The set of all states of nature is denoted by Ω. A statistician will take some action **a** or make a decision **a** if he/she finds out that the state of nature is θ. The set of all actions or decisions is denoted by A. Prior to taking some action, a statistician conducts an experiment for the purpose of collecting data on the state of nature. Experiment outcome is a random variable whose probability law depends on an unknown parameter. If in one experiment a random variable C takes the value x, a statistician takes an action, i.e. makes a decision $a = d(x)$. Function $d(x)$ is called a decision-making function. By this function decision $a = d(x)$ is unambiguously assigned to every experiment outcome x.

Example 1.

A statistician is told that a coin is either normal or double-headed. A statistician cannot examine the coin but he/she can notice what comes up after tossing the coin. Then, a statistician should decide whether the coin is double-headed or not. If a statistician makes a wrong decision, he/she should pay a fee in the amount of 1 CU[15]. In case of a right decision there is no fee (or reward).

Ignoring the fact that a statistician observes one tossing, the problem can be considered as the following game

[15] CU - currency unit

Statistician

(Player A)

$$
\begin{array}{cc|c|c|}
 & & a_1 & a_2 \\
\hline
\text{Nature} & \theta_1 & 0 & 1 \\
\hline
\text{(Player B)} & \theta_2 & 1 & 0 \\
\hline
\end{array}
$$

where

$\theta_1 =$ "state of nature" is that the coin is double-headed

$\theta_2 =$ "state of nature" is that the coin is balanced

$a_1 =$ decision made by a statistician is that the coin is double-headed

$a_2 =$ decision made by a statistician is that the coin is balanced

However, a statistician knows what happened at tossing, i.e. a statistician knows whether a random variable X took the value $x = 0$ (heads) or $x = 1$ (tails). A statistician wants to use that information for making a choice between a_1 and a_2, so that the decision-making function determining which action to take in any of the cases is

$$d_1(x) = \begin{cases} a_1, \, when \;\; x = 0 \\ a_2, \, when \;\; x = 1 \end{cases},$$

i.e. $d_1(0) = a_1 \;\; and \;\; d_1(1) = a_2$.

The purpose of indices is to distinguish different functions. Other possible decision-making functions might be

$$d_2(0) = a_1 \text{ and } d_2(1) = a_1$$

i.e., always choose a_1 regardless of the experiment outcome,

$$or \;\; d_3(0) = a_2 \;\; and \;\; d_3(1) = a_2$$
$$or \;\; d_4(0) = a_2 \;\; and \;\; d_4(1) = a_1$$

The following table gives corresponding values of the loss function

		Statistician	
		a_1	a_2
Nature	θ_1	$L(a_1,\theta_1)$	$L(a_2,\theta_1)$
	θ_2	$L(a_1,\theta_2)$	$L(a_2,\theta_2)$

One option is to select a_1 when $x=0$ and a_2 when $x=1$, which might be expressed as

$$R(d_1,\theta_j)=E\left[L\left(d_1(x),\theta_j\right)\right],$$

whereby $\left[L\left(d_1(x),\theta_j\right)\right]$ is a loss function in case of the decision $d_1(x)$ and Nature strategy θ_j.

Expectation is calculated with respect to a random variable x and

$$\text{with } \theta_1 \quad P(x=0)=1$$
$$P(x=1)=0$$

$$\text{with } \theta_2 \quad P(x=0)=1/2$$
$$P(x=1)=1/2$$

The aforementioned gives

$$R(d_1,\theta_1)=1\,L(a_1,\theta_1)+0\,L(a_2,\theta_1) \quad = 1\times 0+0\times 1 \quad = 0$$
$$R(d_1,\theta_2)=1/2\,L(a_1,\theta_2)+1/2\,L(a_2,\theta_2) \quad = 1/2\times 0+1/2\times 0 \quad = 1/2$$
$$R(d_2,\theta_1)=1\,L(a_1,\theta_1)+0\,L(a_1,\theta_1) \quad = 1\times 0+0\times 0 \quad = 0$$
$$R(d_2,\theta_2)=1/2\,L(a_1,\theta_2)+1/2\,L(a_1,\theta_2) \quad = 1/2\times 1+1/2\times 1 \quad = 1$$
$$R(d_3,\theta_1)=1\,L(a_2,\theta_1)+0\,L(a_2,\theta_1) \quad = 1\times 1+0\times 1 \quad = 1$$
$$R(d_3,\theta_2)=1/2\,L(a_2,\theta_2)+1/2\,L(a_2,\theta_2) \quad = 1/2\times 0+1/2\times 0 \quad = 0$$
$$R(d_4,\theta_1)=1\,L(a_2,\theta_1)+0\,L(a_1,\theta_1) \quad = 1\times 1+0\times 0 \quad = 1$$
$$R(d_4,\theta_2)=1/2\,L(a_2,\theta_2)+1/2\,L(a_1,\theta_2) \quad = 1/2\times 0+1/2\times 1 \quad = 1/2$$

Let us note that a 4×2 game is a zero-sum game for two players where payoffs are equal to corresponding values of the risk function.

Statistician

		d_1	d_2	d_3	d_4
Nature	θ_1	0	0	1	1
	θ_2	1/2	1	0	1/2

Note that d_1 dominates over d_2, and d_3 dominates over d_4, so that d_2 and d_4 can be rejected.
The game is reduced to a 2´ 2 zero-sum game for two players.

Statistician

	d_1	d_3	
Nature	0	1	θ_1
	1/2	0	θ_2

An optimal strategy for a statistician is with probabilities $2/3$ and $1/3$ for d_1 and d_3, respectively.

3. Decision-making criteria

Generally speaking, it is possible to find the best decision-making function only with respect to some criteria. In the sequel we will consider two criteria.

Minimax criterion

According to the minimax criterion, a decision-making function is chosen for which $R(d,\theta)$, maximized by θ, is minimal.

If we apply the minimax criterion to the example from Section 2 with d_2 and d_4, maximal risk for d_1 is $1/2$, and for d_3 it is 1. Thus, d_1 minimizes maximal risk.

Bayesian criterion

If Q is observed as a random variable, in accordance with the Bayesian criterion, a decision-making function is chosen for which $E\left[R(d,\theta)\right]$ is minimal, where expectation is calculated with respect to Q. For the criterion it is necessary to consider Q as a random variable with the given distribution.

Application of the Bayesian criterion to the example from Section 2 requires probabilities to be assigned to strategies of the Nature θ_1 and θ_2. If $P(\theta_1) = p$ and $P(\theta_2) = 1 - p$, then the Bayes risk is for

$$d_1 = 0 \cdot p + 1/2(1-p) = 1/2(1-p)$$
$$d_3 = 1 \cdot p + 0(1-p) = p.$$

When $p > 1/3$, Bayes risk as to d_1 is less than Bayes risk for d_3, so that d_1 is preferred over d_3.
When $p < 1/3$, Bayes risk as to d_3 is less than Bayes risk for d_1, so that d_3 is preferred over d_1.
When $p = 1/3$, both Bayesian criteria are equal and any d_1 and d_3 can be chosen.
The aforementioned can be improved such that the problem is translated to the basic set $\leftrightarrow$ sample, so that decision-making is of the form:

a) A set of all possible values is defined that can be taken by Q in the problem under consideration. This set is denoted by Ω and it is called parameter space.
b) A set of all possible actions or decisions is defined, which can be reached in the problem under consideration. This set is called action space or decision space and it is denoted by A.
c) We define function

$$a = d(x_1, x_2, ..., x_n)$$

of a random sample $x_1, x_2, ... x_n$, which is called a decision-making function.. A set of values of this function is action space A. This means that a statistician takes an action a, if for the realized value of the sample $x_1, x_2, ... x_n$ he/she has obtained $a = d(x_1, x_2, ..., x_n)$.

Example 2. Let $f(x;\mathrm{Q})$ be the Gaussian probability distribution law[16]

$$f(x;\mathrm{Q}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\Theta)^2}{2}}, -\infty < x < +\infty,$$

where Q is any real number.
Parameter space Q is a set of real numbers

$$\Omega = \{\mathrm{Q}½ \infty < \mathrm{Q} < +\infty\}.$$

[16] Sarapa, N.; (1988), p.265, Teorija vjerojatnosti., Zagreb, Školska knjiga, UDK 519.21(075.8)

If for different values of Q we take different actions, the set of actions A is also a set of real numbers

$$\mathrm{A} = \left\{ \mathrm{a}\text{½}\, \infty < \mathrm{a} < +\infty \right\}.$$

For a decision-making function we can take any function of the random sample $x_1, x_2, \ldots x_n$ taken from this normal arrangement. For example, we can take the arithmetic mean[17] of the sample

$$d\left(x_1, x_2, \ldots x_n\right) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

For the realization of sample $x_1, x_2, \ldots x_n$ we obtain

$$\mathrm{a} = d\left(x_1, x_2, \ldots x_n\right) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Loss function

In the game against the nature, a statistician will have gain or loss, depending on the action taken i.e. decision made. Since the action

Table 1:

	possible actions a_1	a_2
states Θ_1	0	10
of nature Θ_2	10	0

In order to find something out about the state of nature, a statistician conducts an experiment whose outcomes are values of a random variable x. Let a random variable X take the value $x = 0$ in case of heads, and the value $x = 1$ in case of tails. In order to use the information obtained through the experiment, a statistician must define all possible decision-making functions.

[17] Crnjac D., Crnjac M.; (2004), Nejednakosti, razlike i odnosi među nekim statističkim sredinama, Zbirka radova XIII., Dane Kordić, Frano Ljubić, Ivan Pavlović, Brano Markić, Ante Markotić, Dražena Tomić, Sanja Bijakšić(ur), pp.171-186, UDK 33 (06.055.2), Ekonomski fakultet u Mostaru, Sveučilište u Mostaru

One possibility is to take action a_1 when $x = 0$ and action a_2, when $x = 1$, that defines the decision-making function

$$d_1(x) = \begin{cases} a_1, \ when\ x = 0, \\ a_2, \ when\ x = 1, \end{cases}$$

whose values are $d_1(0) = \mathrm{a}_1 \ and\ d_1(1) = \mathrm{a}_2$.

The other possibility is to take action a_1 in both cases, so that the decision-making function is

$$d_2(x) = \begin{cases} a_1, \ when\ x = 0, \\ a_1, \ when\ x = 1, \end{cases}$$

with values $d_2(0) = a_1 \ when\ d_2(1) = a_1$.

It is possible to define two more decision-making functions

$$d_3(x) = \begin{cases} a_2, \ when\ x = 0, \\ a_2, \ when\ x = 1, \end{cases}$$

whose values are $d_3(0) = a_2 \ and\ d_3(1) = a_2, \ and$

$$d_4(x) = \begin{cases} a_2, \ when\ x = 0, \\ a_1, \ when\ x = 1, \end{cases}$$

with values $d_4(0) = a_2 \ and\ d_4(1) = a_1$.

As it can be seen, the decision-making function assigns one action to every value of a random variable x. Since we have two values of a random variable x, i.e. $0 \ and\ 1$, and two possible actions a_1 and a_2, there are $2^2 = 4$ decision-making functions that are previously denoted by $d_1(x), \ d_2(x), d_3(x) \ and\ d_4(x)$. Values of these functions can be represented by Table 2.

Table 2:

	d_1	d_2	d_3	d_4
x 0	a_1	a_1	a_2	a_2
x 1	a_2	a_1	a_2	a_1

Choice of the loss function $L(a,\mathrm{Q})$ depends on the nature of the problem under consideration. In problems referring to statistical evaluation, a loss function is taken to be of the form

$$L(a,\mathrm{Q}) = (a - \mathrm{Q})^2,$$

and it is called a quadratic loss error[18]. In problems referring to testing statistical hypotheses of loss functions, function

$$L(a,\mathrm{Q}) = \begin{cases} 0, & \text{if a right decision has been made,} \\ 1, & \text{if a wrong decision has been made} \end{cases}$$

Conclusion

Some optimal strategies, i.e. decision-making functions, are given in this paper. Decision-making criteria as well as criteria for optimizing decision-making functions are also given with respect to some criteria, and certain improvements are made.

REFERENCES

1. Crnjac D., Crnjac M.; (2004), Nejednakosti, razlike i odnosi među nekim statističkim sredinama, Zbirka radova XIII., Dane Kordić, Frano Ljubić, Ivan Pavlović, Brano Markić, Ante Markotić, Dražena Tomić, Sanja Bijakšić(ur), pp.171-186, UDK 33 (06.055.2), Ekonomski fakultet u Mostaru, Sveučilište u Mostaru
2. Fudenberg D., Tirole J.; (1991), Game Theory, MIT Press
3. Gibbons R.; (1992), Game Theory for Applied Economists, Princeton University Press (paperback edition)
4. Martin J.J. Osborne; (2004), An introduction to game theory, Oxford University Press (paperback edition)
5. Myerson R.B.; (1997), p.26, Game Theory Analysis of Conflict, First Harvard University Press (paperback edition)
6. Sarapa, N.; (1988), 85-93, Teorija vjerojatnosti., Zagreb, Školska knjiga, UDK 519.21(075.8)
7. Scitovski R., Galić R. and Šilac-Benšić M.; (1993), p.68, Numerička analiza, vjerojatnost i statistika, Sveučilište J.J.Strossmayera u Osijeku, Elektrotehnički fakultet Osijek, ISBN:953-6032-08-2

[18] Scitovski R., Galić R. and Šilac-Benšić M; (1993), p. 68, Numerička analiza, vjerojatnost i statistika, Sveučilište J.J.Strossmayera u Osijeku, Elektrotehnički fakultet Osijek, ISBN:953-6032-08-2